OCR Further Maths A-level

Additional Pure

Formula Sheet

Provided in formula book	Not provided in formula book

Sequences and Series

Behaviour of Sequences

Periodic	Terms of the sequence repeat regularly. The number of repeated terms is called the period.	$S = \{u_1, u_2, u_3, \dots, u_{n-1}, u_n, u_1, u_2, \dots\}$ Periodic with period n
Oscillating	Periodic with two terms.	$S = \{u_1, u_2, u_1, u_2, \dots\}$
Convergent	Terms of the sequence get closer to a limiting value.	$S = (u_n)$ $\lim_{n\to\infty} u_n = k$
Divergent	Sequence is not convergent, and the sum of the sequence is not finite.	$S = (u_n)$ $\lim_{n\to\infty} u_n$ does not exist $\sum_n u_n$ is undefined
Monotonic Increasing (or Decreasing)	Each term in the sequence is greater/less than or equal to the previous term	$S = (u_n)$ $u_n \geq u_{n-1}$ – monotonic increasing $u_n \leq u_{n-1}$ – monotonic decreasing

Fibonacci and Related Numbers

Fibonacci Recurrence Relation	$u_{n+2} = u_{n+1} + u_n,\ u_1 = 1, u_2 = 1$ Begins 1, 1, 2, 3, 5, 8, …
Golden Ratio	Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$ is the limit of the ratio of consecutive terms in the Fibonacci sequence.
Lucas Recurrence Relation	$u_{n+2} = u_{n+1} + u_n,\ u_1 = 1, u_2 = 3$ Begins 1, 3, 4, 7, 11, 18, …

Solving Recurrence Relations

1st order linear recurrence relations with constant coefficients	$u_{n+1} = ku_n + f(n)$
Homogeneous 1st order linear recurrence relation	$f(n) = 0$ so, of the form $u_{n+1} = ku_n$.
Complementary function	Solution to homogenous version of the recurrence relation. 1st order linear will have the form $u_n = A \times r^n$.
Particular solution	Any solution of the recurrence relation.
General solution	Sum of the complementary function and the particular solution.
Recurrence system	Consists of a recurrence relation, initial conditions, and the range of the variable n.
2nd order linear recurrence relations with constant coefficients	$u_{n+2} = k_1u_{n+1} + k_2u_n + f(n)$
Homogeneous 2nd order linear recurrence relation	$f(n) = 0$ so, of the form $u_{n+2} = k_1u_{n+1} + k_2u_n$
Auxiliary/characteristic equation	Equation obtained after substituting $u_n = r^n$ into 2nd order homogenous recurrence relation and dividing through by r^n.

Roots of auxiliary equation	**Form of complementary function**
Real and distinct roots r_1 and r_2	$Ar_1^n + Br_2^n$
Repeated roots r	$(A + Bn)r^n$
Complex roots z_1 and z_2	$Az_1^n + Bz_2^n$

Number Theory

Divisibility Tests

Divisible by 2	Last digit divisible by 2.
Divisible by 3	Sum of digits divisible by 3.
Divisible by 4	Number formed by final 2 digits divisible by 4.
Divisible by 5	Final digit is 0 or 5.
Divisible by 8	Number formed by final 3 digits divisible by 8.
Divisible by 9	Sum of digits divisible by 9.
Divisible by 11	Result of adding and subtracting digits in alternating order beginning at leftmost digit is divisible by 11.

Division Algorithm

If a is divided by b, where $0 < b < a$, then there is a unique quotient q and residue/remainder r (with $r < b$) such that $a = bq + r$. If $r = 0$, then $b|a$.

Finite (Modular) Arithmetic

If $a = nq + r$ then $a \equiv r \pmod{n}$

Rules	If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then:
$a + c \equiv b + d \pmod{n}$	$a - c \equiv b - d \pmod{n}$
$ac \equiv bd \pmod{n}$	$a^k \equiv b^k \pmod{n}$

Linear Congruences

Linear congruence	Equation of the form $ax \equiv b \pmod{n}$.
Condition for a solution	$d\|b$ where d is the highest common factor of a and n. So if n is prime then $ax \equiv b \pmod{n}$ will have a solution as $hcf(a, n) = 1$ and $1\|b$ for all integers b.
Solutions	$x_1 + \frac{n}{d} \times r$ where x_1 is a solution found by inspection and $r = 0, 1, \ldots, d-1$.

Quadratic Residues

If the congruence $x^2 \equiv q$ has a solution, then q is a quadratic residue $\pmod{n}$.

Prime Numbers

Prime number	An integer greater than 1 with no divisors other than 1 and itself.
Composite number	An integer with at least one divisor other than 1 and itself.
Coprime (relatively prime)	Two or more integers are coprime if 1 is their only common factor.
Fundamental theorem of arithmetic	Every integer greater than 1 is either prime or the unique product of primes (ignoring rearrangements).

Useful results	For integers a, b, c:
If a and b are coprime and $a\|c$ and $b\|c$, then $ab\|c$.	If $a\|b$ and $c\|d$, then $ac\|bd$.
If $a\|b$ and $b\|c$, then $a\|c$.	If $a\|b$ and $a\|c$, then $a\|(bx + cy)$ where x, y are integers.
$hcf(a, b)$ can be found by finding the smallest integer that can be written as $bx + cy$.	If $hcf(a, b) = 1$, then a and b are coprime.

Euclid's Lemma

Euclid's Lemma	If a prime number p divides into the composite number $a_1 \times a_2 \times ... \times a_n$ then p must divide into at least one of a_1 to a_n.
Result from Euclid's Lemma	If $a\|bc$, where a and b are coprime, then $a\|c$.

Fermat's Little Theorem

Form 1	If p is prime and $hcf(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$.
Form 2	If p is prime, then $a^p \equiv a \pmod{p}$.
Warning about pseudo-primes	The converse does not hold. There are pseudo-primes x to base a such that $a^{x-1} \equiv 1(\text{mod } x)$ but x is a composite number.

The order of a modulo p

The order of a modulo p	The smallest positive integer n such that $a^n \equiv 1 \pmod{p}$, $(\gcd(a, p) = 1$ and $a \neq 1$. Notice that this is not necessarily $p - 1$.

Binomial Theorem

$$(a + b)^p \equiv a^p + b^p \pmod{p}, \text{ where } p \text{ is prime}$$

Groups

Binary Operations

Binary operation	A process involving two members of a set.

Definitions	Consider elements a and b of a set S.
Closed	A set is closed under an operation $*$ if for all $a, b \in S$, $a * b \in S$.
Commutative	The operation $*$ is commutative if for all $a, b \in S$, $a * b = b * a$.
Associative	The operation $*$ is associative if for all $a, b \in S$, $(a * b) * c = a * (b * c)$.
Identity element e for the operation $*$	$e \in S$ satisfies: $a * e = e * a = a$ for all elements $a \in S$.
Inverse a^{-1} for element a with operation $*$	$a^{-1} \in S$ satisfies: $a * a^{-1} = a^{-1} * a = e$ where e is the identity element.
Self-inverse a^{-1}	$a^{-1} \in S$ satisfies: $a^{-1} = a$ so $a^2 = e$ where e is the identity element.

Definition of a group

Conditions for a set to be a group under operation $*$
Closed
Associative
The set contains an identity element e
Every element of the set has an inverse

Abelian Group	If all elements of the group commute under the binary operation.

Orders and Elements of Groups

Order of a group, $\|G\|$	The number of elements the group contains.
Order of an element	The smallest power an element is raised to that gives the identity element.

Subgroups

Subgroup	H is a subgroup of the group G if H is a subset of G and H is also a group under the same binary operation.
Trivial subgroup	The trivial subgroup is $\{e\}$ where e is the group identity element.
Proper subgroup	A subgroup of G which is not G itself.

Cyclic groups

Cyclic groups	Every element of the group G is of the form a^n, where $a \in G$ and $n \in \mathbb{Z}$. Notice that a is called the generator of the group and is not necessarily unique.

Properties of cyclic groups
Commutative
At least one element of the group must be order n

Properties of Groups

Order of group is 1	Group is $\{e\}$.
Order of group is 2,3,4,5 or 7	Group is cyclic.
Order of group is 4	Group is cyclic where: at least one element has order 4 or group is Klein group.
Order of group is 6	Group is cyclic if one element has order 6, otherwise group forms a symmetric group.

Lagrange's Theorem

The order of a subgroup H is a factor of the order of the group G.

Implications
The order of each element of G is a factor of the order of G.
If G has an order which is prime, then G has no proper subgroups.
If H has an order p, where p is prime, then every non-identity element of H has order p.

Isomorphism

Two groups G and H are isomorphic if they have the same structure and have a one-to-one correspondence between elements

Further Vectors

Vector Produc

$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta\, \hat{\boldsymbol{n}}$, where $\boldsymbol{a}, \boldsymbol{b}, \hat{\boldsymbol{n}}$, in that order, form a right hand triple.

Observations	
Magnitude	$\lvert\boldsymbol{a} \times \boldsymbol{b}\rvert = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert \sin\theta$
Condition for parallel or co-linear vectors	$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$ given that $\boldsymbol{a} \neq \boldsymbol{0}$ or $\boldsymbol{b} \neq \boldsymbol{0}$
Not commutative	$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
Distributive over addition	$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$
Linear	$\boldsymbol{a} \times \lambda\boldsymbol{b} = \lambda\boldsymbol{a} \times \boldsymbol{b} = \lambda(\boldsymbol{a} \times \boldsymbol{b})$
Equation of a straight line	$(\boldsymbol{r} - \boldsymbol{a}) \times \boldsymbol{d} = \boldsymbol{0}$

Properties of scalar triple product $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$	
Vectors moving in cyclic order	$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b}$
For $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ to be co-planar	$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = 0$
If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel or collinear	$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = 0$ since $\boldsymbol{a} \times \boldsymbol{b} = 0$
$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = det \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$	

Area of triangle with sides $\boldsymbol{a}, \boldsymbol{b}$.	$\frac{1}{2}\lvert\boldsymbol{a} \times \boldsymbol{b}\rvert$
Area of parallelogram with sides $\boldsymbol{a}, \boldsymbol{b}$	$\lvert\boldsymbol{a} \times \boldsymbol{b}\rvert$
Volume of tetrahedra with base with sides $\boldsymbol{a}, \boldsymbol{b}$ and adjacent side $\boldsymbol{c}$	$\frac{1}{6}\lvert(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}\rvert$
Volume of parallelopipeds with base with sides $\boldsymbol{a}, \boldsymbol{b}$, adjacent side $\boldsymbol{c}$ and height $\lvert\boldsymbol{c}\rvert \cos\theta$.	$\lvert(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}\rvert$

Surfaces and Partial Differentiation

Partial Differentiation

Mixed derivative theorem	$f_{xy} = f_{yx}$ for suitably well-defined continuous functions f.

Stationary Points

Stationary points of a function $f(x, y)$ occur when $f_x = f_y = 0$. There are three types of stationary points: maximum, minimum or saddle.

For 3-D surfaces given in the form $z = f(x, y)$ the Hessian Matrix is given by

$$\boldsymbol{H} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Local minimum	$\lvert\boldsymbol{H}\rvert > 0$ and $f_{xx} > 0$
Local maximum	$\lvert\boldsymbol{H}\rvert > 0$ and $f_{xx} < 0$
Saddle point	$\lvert\boldsymbol{H}\rvert < 0$
Inconclusive	$\lvert\boldsymbol{H}\rvert = 0$

Tangent Planes

The equation of a tangent plane to the curve at a given point $(x, y, z) = \big(a, b, f(a, b)\big)$ is $z = f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b)$

Further Calculus

Reduction Formulae

$I_n = \int \sin^n x \, dx$	Write as $\int \sin x \times \sin^{n-1} x \, dx$
$I_n = \int \tan^n x \, dx$	Write as $\int \tan^2 x \times \tan^{n-2} x \, dx$

Arc Lengths and Surface Areas

Cartesian arc length	$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$
Parametric arc length	$\int_a^b \sqrt{\dot{x}^2 + \dot{y}^2} \, dt$
Cartesian surface area of revolution about the x −axis	$2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$
Cartesian surface area of revolution about the y −axis	$2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$
Parametric surface area of revolution about the x −axis	$2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$
Parametric surface area of revolution about the y −axis	$2\pi \int_c^d x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$